The LS-TaSC™ Multipoint Method for Constrained Topology Optimization

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Abstract

The new multi-point constrained optimization scheme is for the constrained topology design of highly nonlinear structures for which analytical design sensitivity information is hard to compute. These highly nonlinear structures are designed for multiple load cases and multiple constraints, which means that the final design should have load paths for each load case as well as satisfy the constraints. This is done here by using two sets of variables: the local variables describing the part topology on the element level and the global variables consisting of the load case weights and part masses. The two sets of variables are treated differently in the design algorithm: the local variables are computed using a suitable method such as fully stressed design, while the values of the global variables satisfying the constraints are computed using numerical derivatives and mathematical programming.

Introduction

The topology design of structures [1,2] has been investigated for several decades, especially for linear problems. The goal is that of finding the topology of a structure supporting the required load by starting off with a ground structure within which the required structural topology or load path must be found. The ground structure is meshed finely using finite elements and typical design variables are the amount of material within each finite element.

Normally, constrained design optimization problems are solved considering the design sensitivity information (the derivatives of the constraints and objective with respect to the design variables). For highly nonlinear mechanics, such as crash analysis, it is prohibitively expensive to implement this design sensitivity analysis into the analysis code. So for the case of highly nonlinear structural problems topology optimization is done using methods such as Hybrid Cellular Automata [3,4], Evolutionary Topology Optimization [5], or other heuristics such as Prescribed Plastic Strain/Stress [6] – all methods which do not require design sensitivity information.

The use of design sensitivity analysis in highly nonlinear problems is very rare, but not non-existent: Peterson [7], for example, used 2D beam elements and rigorously computed the sensitivities for crashworthiness design. But the high development cost makes it very doubtful that rigorously computed sensitivities will ever be available in general purpose codes. An alternative to the design sensitivity analysis of highly nonlinear structures is the use of metamodels or other approximations [8] if the number of variables is small (say order of ten or less). This approach is therefore not feasible for topology design due to the millions of variables used. If the number of variables can be reduced – for example, by using a reduced basis formulation [9] – then metamodels can be used, and this approach is indeed used in shape optimization by using a small number of shape variables define using a mesh morphing software [10] to control the many nodal locations.

Another method of working around the lack of design sensitivity information in nonlinear problems is the use of equivalent static loads [11], in which static loads which reproduce the same displacement field as the dynamic loads are used. This allows the use of a linear model for the design optimization. It is accordingly required that FE models are created for both the nonlinear analysis and the linear analysis together with a system passing the data from the one to the other. Industrial users [12] report that the method, in its current form, is promising for nonlinear problems with moderate deformations.

This study considers the variables in two sets. Variables can be separated into sets if there is no significant coupling between them. For example, the layout of a linear structure is dependent on the location of the loads and supports, while the mass only depends on the magnitude of the load – one can therefore compute the layout and mass separately. This can also be observed in nature: if you look at the design of a muscle, then the size of the muscle is largely dependent on the load, while the design of the individual muscle fibers is a result of evolution and chemistry, so the muscle size design and muscle fiber design can be pursued as independent design questions. In this paper the two sets of design variables are also computed using different design methodologies, which allows the use of finite differences or metamodels to compute the design sensitivity information with respect to a small, but important set of global variables, while the very large set of local variables is solved for using a fully stressed design approach.

The one set of variables is the local variables, which are the normal topology density variables describing the amount of material in an element. The algorithm used for the design of the local variables is a differentiable implementation of the Optimality Criteria method as modified for dynamic problems. The basis of the method is the optimality criterion of having a constant internal energy density in the elements and with the variables updated as for the fully stressed algorithm [9]. More information on the suitability of the internal energy density as a design criteria for the design of highly nonlinear structures is given by Öman and Nilsson [13]. The initial modifications of the algorithm for use in highly nonlinear problems are related to the work of Patel [4]. A major difference from Patel's work is that a cellular algorithm, as introduced by Tovar [3], is not used. Instead the design field is filtered using a filter similar to the one described by Bendsøe and Sigmund [2]. The resulting method were further modified based on feedback of customers using it for highly nonlinear problems and to allow constrained optimization using finite differences or metamodels.

The other set of variables is the global variables consisting of the part mass fractions and load case weights. In the past gradient-free methodologies have used these global variables to satisfy the constraint bounds for highly nonlinear problems. Specifically, Patel [3] used the part mass fraction to satisfy a constraint, Bandi *et al.* [15] used the load case weights to control force displacement behavior, and Aulig *et al.* [16] used a linear combination of the load case weights to search for a multi-objective solution of multi-disciplinary topology optimization problem. The adjustments to the part mass fractions and load case weights by Patel [3] and Bandi *et al.* [15] was done by creating an error signal as in control theory and using some heuristics to reduce this error signal. This approach is valuable in some cases, but it requires the analyst to define this heuristic. An automated design approach using mathematical programming is preferred, which then in turn requires the computation of derivatives.

Global and Local Variables

In this study the design variables are considered in two groups and the best design process is used for each group. The variables are partitioned into:

- The **local variables** $\boldsymbol{x}$. These are the classical topology variables describing the layout or local design of the structure; for example, the amount of material in an element. These variables therefore describe the load paths in the structure.

- The **global variables** ξ. These describe a global quantity such as the mass fraction of a part, or a load case weight. These global variables are set to satisfy the constraints, and the derivatives of the constraints with respect to the global variables are computed using a multipoint method such as finite differences or metamodels.

If the global variables are chosen such that $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi})$, then the relationship between a response R and the variables can be written as

$$R = R(\left(\boldsymbol{x}(\boldsymbol{\xi})\right))$$

which allows the numerical computation of the derivative of R with respect to ξ.

The designer should of course model the problem such that the global variables will solve his design optimization problem. The pitfall is thinking that a large number of local variables will result in multiple active constraints, while the local variables will not do this in this approach; it is the choice of global variables that enables multiple active constraints. Specifically, if the design problem requires multiple constraints to be active, then the default situation of a single part subject to a single load case is too simple. Designing a single part subject to a single load case will result in only one active constraint; to have an additional constraint active, the designer must divide the design part into two design parts, or add another design part, or add another load case. In general the number of active constraints can be no more than the number of global variables, which is one less than the sum of the number of parts plus the number of load cases.

Optimization Using Global and Local Variables

There are two optimization problems: one for the global variables and one considering the local variables. The optimization problem considering only the global variables **ξ** can be written as:

$$\min_{\boldsymbol{\xi}} \mathrm{f}(\boldsymbol{\xi}) \text{ with } \boldsymbol{\xi} = (M_1, \dots, M_p,\ w_1, \dots, w_L) ,$$

subject to

$$g_i(\boldsymbol{\xi}) < 0 \; with \; i = 1, \dots, m$$
$$\xi_i^L \leq \xi_i \leq \xi_i^U$$

with **ξ** the global variables bounded by $\boldsymbol{\xi}^L$ and $\boldsymbol{\xi}^U$, $\boldsymbol{M}$ the part mass fractions for p parts, $\boldsymbol{w}$ the load case weights for L load cases, and f and **g** the objective function and constraints respectively.

The optimization problem considering the local variables $\boldsymbol{x}$, for each design part, can be written:

$$\min_{\boldsymbol{x}} \sum_{i=1}^{N} \sum_{j=1}^{L} \left(w_j U_{i,j}(x_i) - U^*\right)$$

,

subject to

$$\sum_{i \in part\, k} x_i V_i \;\; / \, V_0^k \;\; = M_k \; with \; k = 1, \dots, p$$

$$0 \leq x_{min} \leq x_i \leq 1.0$$

in which $\boldsymbol{x}$ is the amount of material in each of the N elements, $U_{i,j}$ represents the internal energy density of the i^{th} element for load case j, V_i is the volume of i^{th} element, V_0^k the volume of the ground structure for part k, U^* is the internal energy density set point, $\boldsymbol{M}$ is the vector containing the p part mass fractions, and $\boldsymbol{w}$ is the vector of load case weights for the L load cases. The mass constraint for mass fraction M_k is computed considering only the variables associated with the part k.

The previous equation suggest how further global variables can be synthesized. For example, the set point U^* can have a spatial variation controlled by the global variables, which then introduce in effect global variables controlling the spatial variation of the stiffness. Also, $U_{i,j}$ the internal energy density of the i^{th} element for load case j, is normally taken as the maximum value found over all time steps. The value of $U_{i,j}$ can instead be controlled by a temporal global variable mitigating for example reaction forces.

Note that there are now two objectives, the global objective computed by the global variables, and the local objective used in the local variables computations. A vehicle crash example of this would be a global objective of maximizing the energy absorption of the structure subject to an intrusion constraint, while requiring a local objective of the stiffest structure.

The following sections describe how to solve simultaneously for both the global and local variables. This is likely to be only one of many possible methodologies as suggested by the literature on multidisciplinary optimization [17]. The selection discussed here was made to minimize computational cost.

Iterative Optimization Scheme

An iterative scheme is used in which a sequence of candidate designs is evaluated till convergence. Both the global and local variables are recomputed for each iterate subject to side constraints on the variable values. The global variables are computed from the previous iterate considering the previous global variable values, the constraint values, and the constraint derivatives. The local variables are then computed from the previous iterate using the local design variable values, the new global design variables, and the design field (typically the internal energy density).

The only change to the standard scheme as used by Patel [4] and others is therefore the computation of global variables at each iteration. This can be done using local or midrange approximations to the global variables as described by Barthelemy and Haftka [8]. In this study, Taylor expansions based on the numerical derivatives were used. The Taylor expansion for a function g around a point $\boldsymbol{\xi}_0$ is simply:

$$G(\boldsymbol{\xi}) = g(\boldsymbol{\xi}_0) + \sum_{i=1}^{n} (\xi_i - \xi_{0i}) \left(\frac{\partial g}{\partial \xi_i}\right)_{\xi_0}$$

Using $F(\boldsymbol{\xi})$ and $G_i(\boldsymbol{\xi})$ as the Taylor expansion to $f(\boldsymbol{\xi})$ and $g_i(\boldsymbol{\xi})$, and the move limits $\xi_i^{L\prime}$ and $\xi_i^{U\prime}$, the optimization problem becomes:

$$\min_{\boldsymbol{\xi}} \ \mathrm{F}(\boldsymbol{\xi}) \ \text{ with } \boldsymbol{\xi} = (M_1, \dots, M_p,\ w_1, \dots, w_L)\ ,$$

subject to

$$G_i(\boldsymbol{\xi}) < 0 \ with\ i = 1, \dots, m$$
$$\xi_i^{L\prime} \le \xi_i \le \xi_i^{U\prime}$$

with $\boldsymbol{\xi}$, $\boldsymbol{M}$, p, $\boldsymbol{w}$, L, F and **G** as described previously.

The global variable move limits $\xi_i^{L\prime}$ and $\xi_i^{U\prime}$ are centered around the optimum of the previous iteration and are chosen here as

$$\xi_i^{L\prime} = \xi_i - k\left(1 + e^{-iteration/10}\right)$$
$$\xi_i^{U\prime} = \xi_i + k\left(1 + e^{-iteration/10}\right)$$

with k taken as 0.05 and 0.1 for mass fractions and load case weights respectively. This part of the methodology is therefore reminiscent of the successive response surface method described by Stander and Craig [18], though simpler in its implementation. Indeed, you only need to be familiar with Sequential Linear Programming [9] in order to implement this.

Intermediate variables as described by Barthelemy and Haftka [8] can be used to improve accuracy and performance in the approximations. It is natural to rewrite the load case weights $(w_1, w_2,\ \cdots, w_L)$ as ratios. Here intermediate variables for the weights are constructed by defining them as a ratio to w_1 resulting in the set $(w_1/w_1, w_2/w_1,\ \cdots, w_L/w_1)$, which after setting $w_1 = 1$, becomes $(1, w_2,\ \cdots, w_L)$, which is then simply the elimination of the w_1 variable. The log of these variables is mathematically more stable, which yield the final set of load case weight variables as $(\log(w_2),\ \cdots, \log(w_L))$.

The resulting optimization problem can now be solved for the global variables using any suitable mathematical programming algorithm. Dynamic-Q [19] is used here.

Numerical Derivatives of the Constraints w.r.t. the Global Variables

If there are only a few global variables, then the derivatives of the constraints with respect to the global variables can be computed numerically.

Computing derivatives numerically in its easiest form comprises of (i) perturbing the global variables, (ii) evaluating the structure at these perturbed values of the global variables, and (iii) using the variation in results to estimate the derivatives. This is indeed the approach followed here, but with the added step of computing the values of the local variables from the perturbed values of the global variables. The steps required at each design iteration accordingly are:

- Create the finite differencing stencil containing all of the combinations of perturbed global variable values needed
- For each combination of global variables values in the stencil:
    - Create a design field (see the section on computing the local variables) considering the perturbed load case weights

        - Compute the local variable values considering the design field and perturbed part mass fractions
        - Create an FEA model for the obtained local variable values
        - Analyze this FEA model
        - Extract the constraint values
    - Compute the derivative of the constraints with respect to the global variables using finite differences

Highly nonlinear problems contain, by definition, significant noise. The derivative values are accordingly filtered over the last *n* iterations using exponential decay as

$$\check{v} = \frac{\sum_{j=1}^{n} v_j e^{-k(n-j)}}{\sum_{j=1}^{n} e^{-k(n-j)}}$$

with v_j the value at iteration *j*, and *k*, the decay time constant, taken as 3.0. This is not always required: linear problems are not noisy, and the filtering is accordingly not required.

This is clearly not the only possible multipoint approach. In LS-TaSC we also use the steps above to construct a response surface as well as direct optimization which simply use the global variable values and not their derivatives. It is also possible to set up the process in LS-OPT®.

Examples

The three examples illustrate doing topology design using part mass fractions and load case weights as global variables. The first problem is a linear problem, purposefully simple in order to be easily reproduced by other researchers. The other two problems addresses design concerns raised by industrial users investigating highly nonlinear structures: the design of multiple parts with conflicting effects on a response, and designing for multiple highly nonlinear load cases.

Linear beam with multiple load cases and asymmetrical constraint

The beam structure shown in Figure 1 shows how the global variables are used to satisfy constraints. The structure is subject to two load cases. The two global design variables are the part mass fraction and the weight of the second load case. The structure and load cases are symmetrical, but the constraints are chosen as asymmetrical – the asymmetrical constraints are satisfied through changes to the load case weight.

The mass fraction of the part is minimized. Two displacements, each at the location of the load, are considered:

$$Y1 \geq -0.002$$
$$Y2 \geq -0.004$$

The initial mass fraction for the part is 0.3 and the initial weight of the second load case is 1.0. Using central differences to compute the derivatives with respect to the global variables, four variations of the design must be analyzed per load case, resulting in ten FEA analyses per design iteration.

The resulting design histories are as shown in Figure 2. In the plots it can be seen that the mass fraction variable is approximately constant after iteration 10. The weight variable is therefore responsible for the large decrease in the value of Y1. Note that the weight variable overshoots, which is typical of weight variables – the reason being that the effect of a change in a load case weight manifests over a large number of iterations in the design process.

The design sensitivities are also shown in Figure 2. The derivatives with respect to the part mass fractions shows some increases and decrease, this variation is likely due to the local variables changing the stiffness of the structure by creating holes and strengthening connections. The derivatives with respect to the load case weight show similar increases and decrease at the same design iterations, presumably for the same reason. Additionally, the derivatives with respect to the load case weights shows a decrease in magnitude when the load case weight increases in magnitude, which is expected.

Multiple parts problem

The structure shown in Figure 3 represents the front floor plan of a vehicle. The two design parts are part 3 which is the design of the engine compartment and part 5 which is the design of the passenger compartment. The intrusion into the passenger compartment depends strongly on the mass fraction of part 3 relative to part 5.

Only one load case is used, this load case consists of an applied displacement to the front of the vehicle while the rear of the passenger compartment is fully supported. The initial mass fractions for both part parts are 0.3. Using central differences to compute the derivatives with respect to the global variables, four variations of the design must be analyzed per load case, resulting in five FEA analyses per design iteration.

The mass of the structure is minimized. The intrusion is defined by the difference of the displacements of nodes 987 and 1523 is constrained to be less than 0.003, and the energy absorption of the design parts is constrained to be more than 800 units, with most of the energy absorbed by part 3 – part 3 is required to absorb at least 1.5 times more energy than part 5. The internal energy of the design parts are extracted and named E_3 and E_5 after the relevant part IDs. The constraints are therefore:

$$(d_{987} - d_{1523}) \, / \, 0.003 \ \leq 1$$
$$\frac{E_3}{1.5 \, E_5} \geq 1$$
$$(E_3 + E_5) \, /800 \geq 1$$

The results are as shown in Figure 4. The figure shows that part 3, the engine compartment, was made light to ensure that it absorbs most of the energy – thereby meeting the requirement that it absorbs 1.5 times more energy than the passenger compartment. Also, the two compartments together absorbed the required 800 units of energy. The intrusion constraint is not active.

The design sensitivity information (the derivatives with respect to the part mass fractions) is also reported in Figure 4. This design sensitivity information, used to compute the optimum design shown previously, helps to understand the structural behavior – e.g. how the structure should be changed to obtain a specific behavior. The values contains some noise, specifically relative the mass fraction of part 3, which has few elements, but the magnitude of the noise is small relative to the value of the result. In some cases the results are constant over all iterations, but this is with respect to a different part in each case, so this observation cannot be generalized.

Large displacement problem

This is a large displacement problem as shown in Figure 5. The objective here is to obtain a structure that gives the same maximum displacement if impacted at various locations at the top. This is simulated by two load cases where an impactor hit the structure and the top center and an offset location as shown in the plots. The final structure is therefore required to have the same displacement for two load cases, as well as to be symmetric around the center, thus removing the computational cost of a third load case.

The downward displacements are monitored at the locations of the center and offset impact. These two displacements are required to be the same. The part mass fraction is kept constant at 0.25. The part mass fraction is however kept as a variable together with load case weight of the offset load case in order to compute design sensitivity information. The only constraint is:

$$Disp_{center} = Disp_{offset}$$

For this example the upper and lower bound on the weight variable is taken as $\pm 0.25\left(1 - e^{-iteration/10}\right)$ to ensure that it has a measurable effect during the numerical derivative computations. Using central differences to compute the derivatives with respect to the global variables, four variations of the design must be analyzed per load case, resulting in ten FEA analyses per design iteration.

The final design and deformations for both load cases are shown in Figure 5, while the design histories are shown in Figure 6. It can be seen that the process converges to a design with both displacements equal and that the load weight oscillates over several iterations to enforce this equality of the displacements.

The design sensitivities are shown as well in the history plots. The derivative of the center displacement with respect to part mass fraction has some noise starting at iteration ten; this noise is likely due to the underlying local variables establishing the topology of the part at this stage. Also the derivatives with respect to the part mass increases greatly in magnitude from iteration one to iteration ten; this is like due to both the local variables establishing the best topology and the change in the load case weight. One derivative with respect to the load case weight likewise shows an increase in magnitude to iteration ten; this is also assume to be due to both the local variables and the change in load case weight.

Multi-disciplinary problem

The structure is as shown in Figure 7 is subject to an impact and two linear load cases as well as required to be symmetric around the XY and ZX planes. The objective is the minimum mass of the structure, while the load cases and constraints are:

1. *Impact* An impactor hits the structure as shown in the figure with a constraints of

$$Reaction\ force_{impact} \leq 200e6$$
$$Energy\ absorbed_{impact} \leq 11.2e6.$$

2. *Bending* A linear analysis of the bending load as shown in the figure with a constraint of

$$Displacement_{bending} \leq 0.3125.$$

3. *Torsion* A linear analysis of a torsion load as shown in the figure with a constraint of

$$Displacement_{torsion} \leq 0.075.$$

Note that the reaction force constraint conflicts with the displacement constraints, because the one needs a compliant structure and the other a stiff structure.

The problem has three global design variables: the part mass fraction, the crash load case weight, and the torsion load case weight. Using central differences to compute the derivatives with respect to the global variables, six variations of the design must be analyzed per load case, resulting in twenty-one FEA analyses per design iteration.

The final design is shown in Figure 7, while the design histories are shown in Figure 8. It can be seen that the both the energy absorption and bending displacement constraints are active. The final solution therefore has two active constraints, but three design variables. This is due to either the choice of the move limits for the global variables (the best choice of the move limits is one of the current areas of investigation) or due to the bending load path being good enough for the torsion load case as well (the torsion load case is simply not active).

Conclusion

The results are encouraging for the constrained topology design of structures for which analytical design sensitivity information is not available.

Two sets of variables are used in a simultaneous iterative design process. The global variables, such as the part mass fractions and the load case weights, are used to satisfy the constraints; while the local variables, typically the amount of material in an element, are used to simultaneously compute the load paths. It was shown that constrained, multiple load case topology problems can be solved in this manner.

Numerical derivatives of the constraints with respect to the global variables can be computed. These were used to construct approximations used together with mathematical programming to satisfy the constraints.

The methodology has the advantage of allowing general constraints – a constraint can depend on different parts and load case weights, or the constraint can be a complex computation of results extracted from the FEA analysis.

It should be noted that the results are for a relatively recent and growing method with only two years of research and development completed. With time the results should improve.

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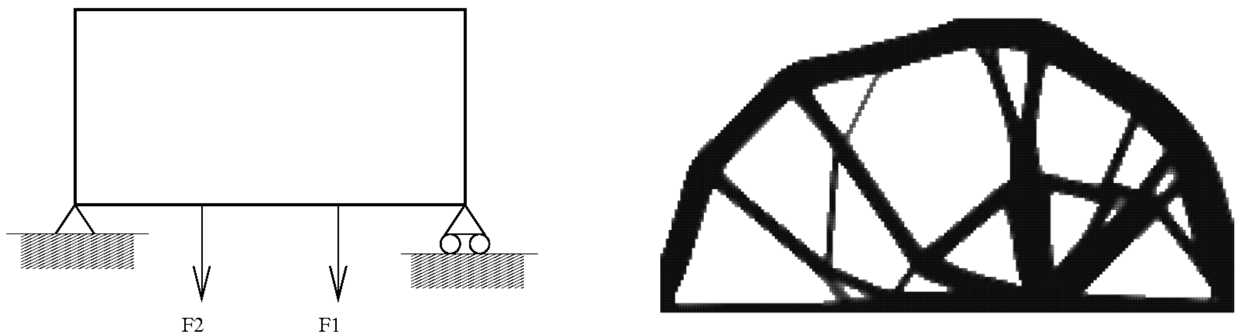

Figure 1: The linear, multiple load case example. The initial ground structure and the loads for the two load cases are shown on the left, while the final design is shown on the right. The final design is asymmetric because the constraints are asymmetric.

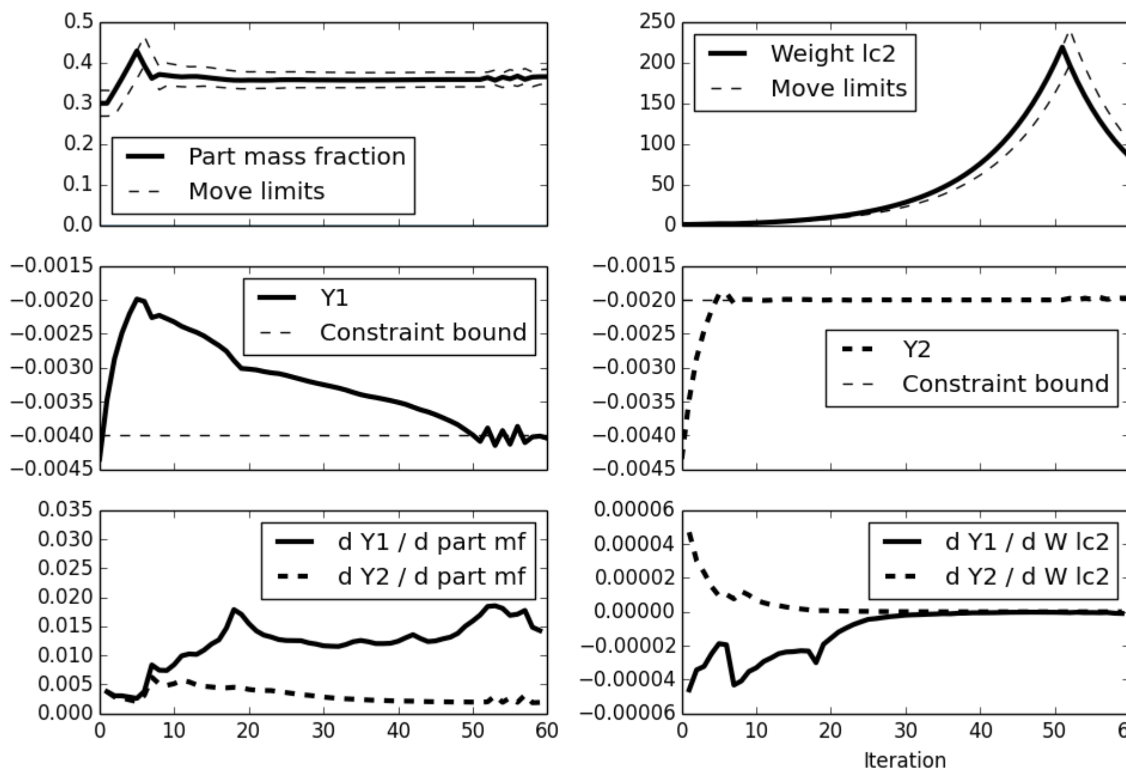

Figure 2 Convergence histories from the linear, multiple load case example. The upper two plots show the histories of the design variables, while the other plots show the constraint values and their derivatives. Together the plots show the role of the weight variable in driving constraint Y1 to the bound.

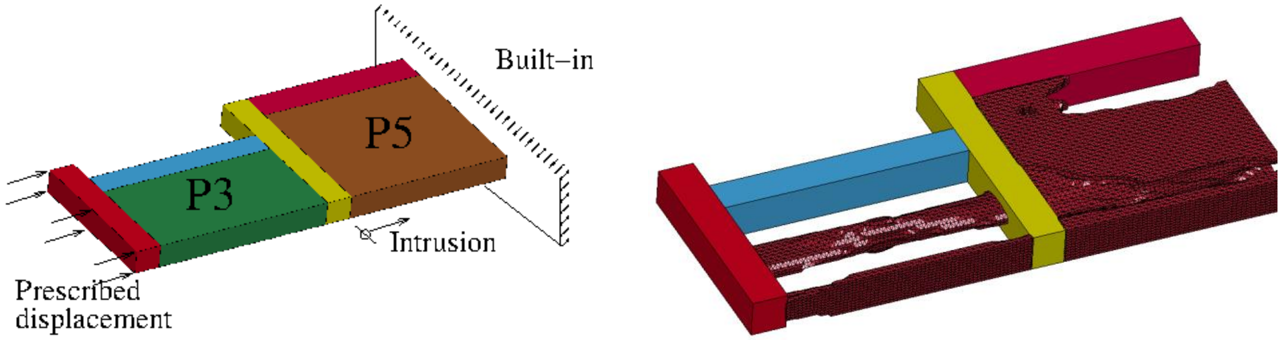

Figure 3: The multiple part example. The geometry and loading conditions are shown on the left with the final design on the right.

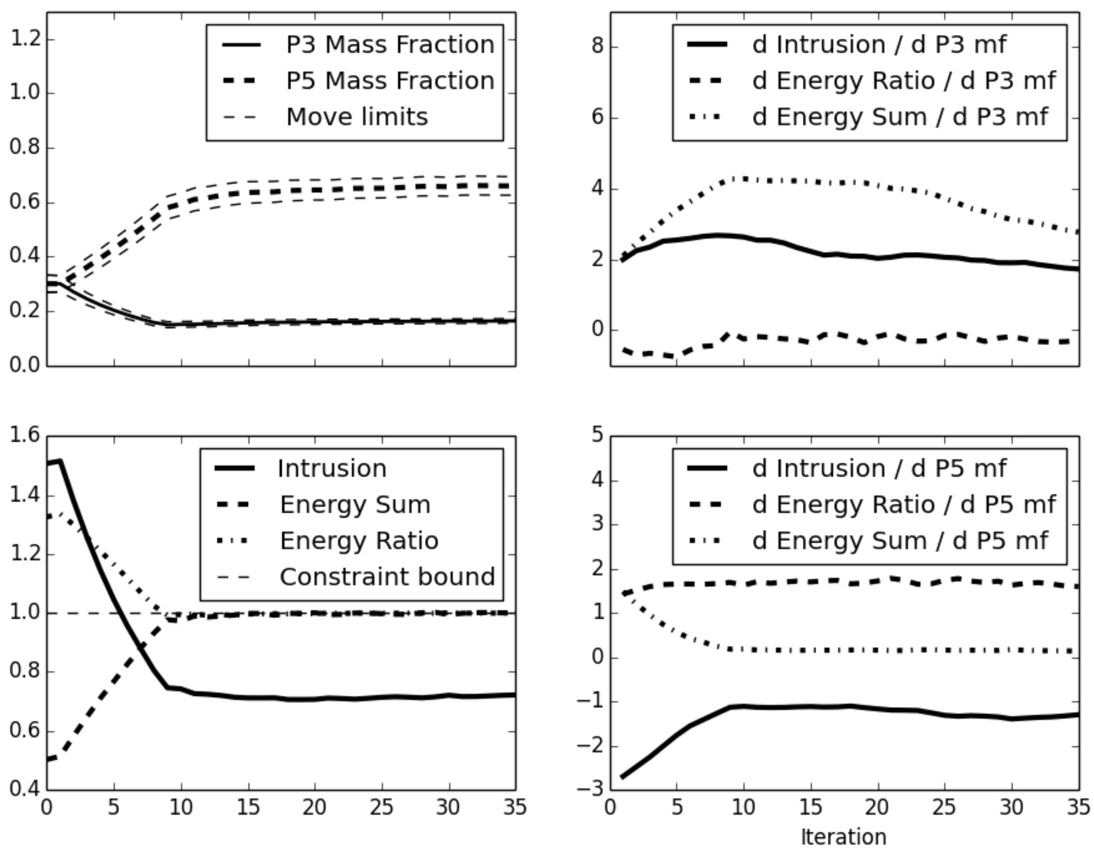

Figure 4 Convergence histories of the multiple part example. The part mass fractions, constraints values, and the derivatives of the constraints are shown.

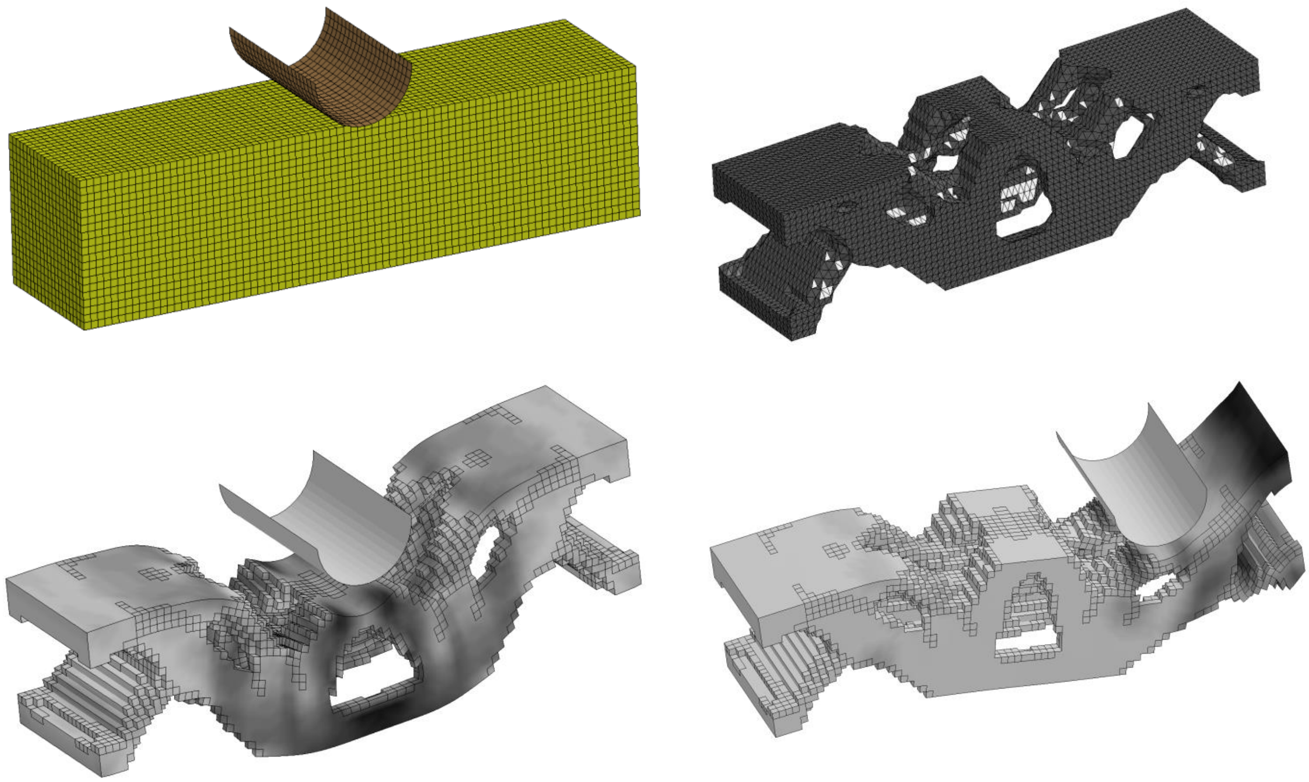

Figure 5 Large displacement problem. The ground structure is shown in the left top, the final design is shown in the right top, while the lower plots show the two load cases in the deformed state.

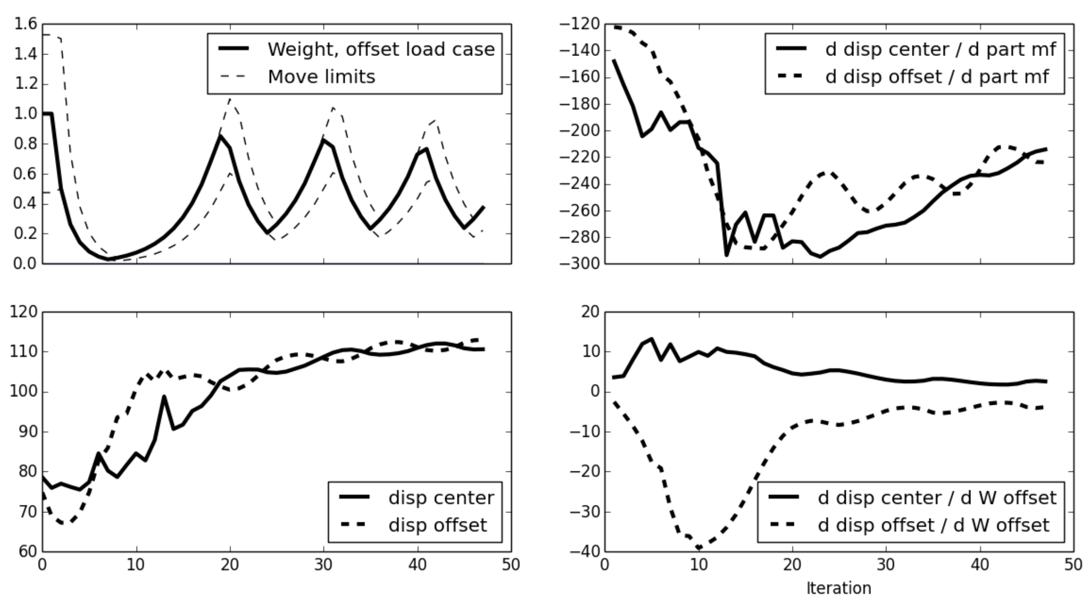

Figure 6 Convergence histories for the large displacement example. The weight variable, constraint values, and derivatives of the constraints are shown.

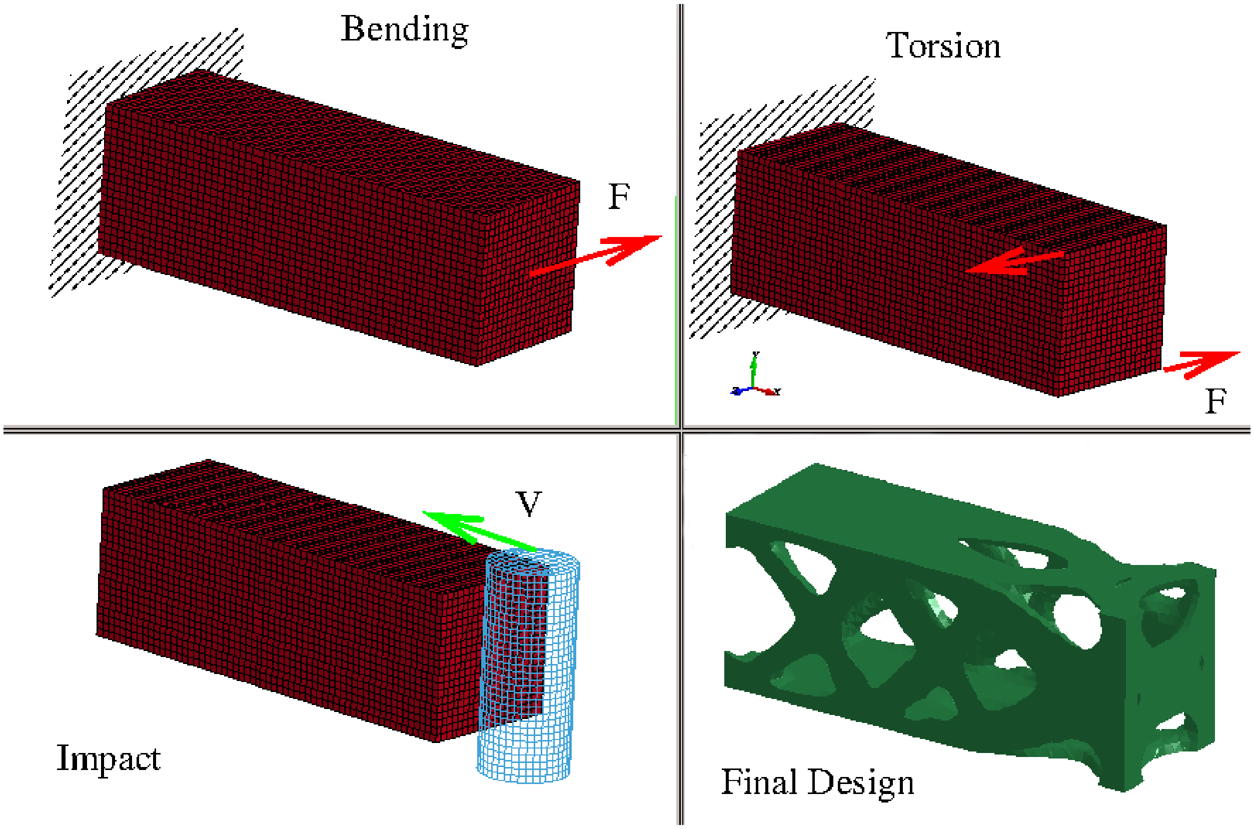

Figure 7 Multi-disciplinary example. The three loading conditions are shown together with the final design on the bottom right.

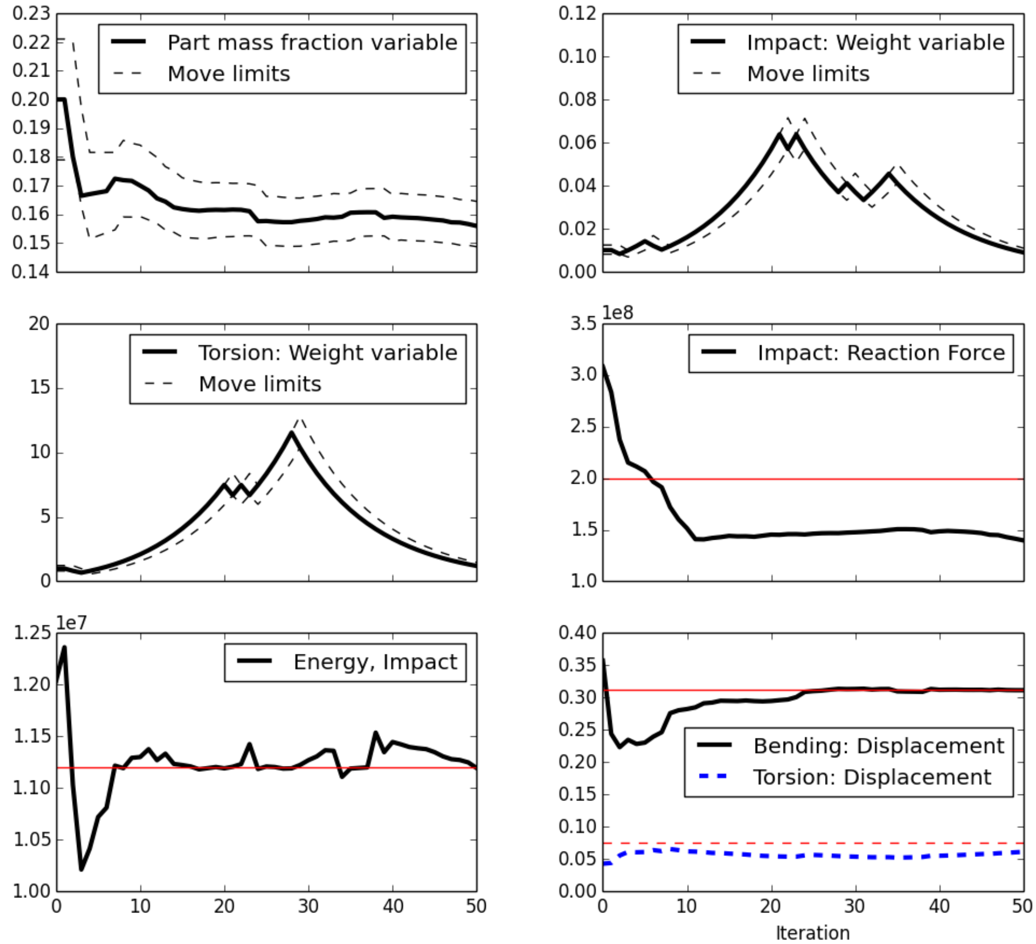

Figure 8 Convergence histories for the multi-disciplinary example. The variables and constraint values are shown. The active constraints are the energy absorption of the crash case and displacement of the bending case.